$$+\sum_{k=1}^{\infty}\left(\frac{\zeta}{z}\right)^k \sum_{j=0}^{\infty} \frac{A_{m+2j} B_{r+j,\, j+q+k} (z\zeta)^{2j+k}}{\Gamma(2j+\gamma+1)\,\Gamma(2j+2k+\nu+1)}\Bigg\}$$

In applying the mentioned procedure to the obtained particular solutions the relationships to which the quantities $a_{k,s}$ and $b_{k,s}$ in (1.5) are subject must be kept in mind

$$a_{r,\, r+k}(\delta) = \alpha_1^{-2r} \varkappa_{2r,\, 2r+k}, \qquad b_{r,\, r+k}(\delta) = \alpha_1^{-2r-1} \varkappa_{2r+1,\, 2r+k+1} \tag{8.2}$$

where

$$\varkappa_{r,\, r+k} = \sum_{j=0}^{r} 2^{2r-2j}\left(-\frac{2}{\delta+1}\right)^j \frac{(r+k)!\,(2j+2k)!}{j!\,(j+k)!\,(j+2k)!\,(r-j)!}\,, \qquad \alpha_1 = 1-\alpha = \frac{2}{1+\delta}$$

In conclusion, let us note that the parameter $0 \leqslant \delta \leqslant \infty$. If the radii of shell curvature are $R > 0$, $R_1 > 0$, $(R_1 \geqslant R)$, then $1 \leqslant \delta \leqslant \infty$; if $R > 0$, $R_1 < 0$ and $|R_1| \geqslant |R|$, then $0 \leqslant \delta \leqslant 1$. The values $\delta = 1$ and ∞ correspond to cylindrical and spherical shells. The value $\delta = 0$ corresponds to a shell of hyperbolic type for $|R| = |R_1|$ (pseudosphere).

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OPTIMAL STRATEGIES IN A LINEAR DIFFERENTIAL GAME

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The game problem of bringing onto a prescribed set a controlled object whose motion is described by linear differential equations is considered. The conditions under which a saddle point exists in the class of generalized strategies in the differential game under investigation are derived. A procedure for constructing the players' generalized optimal strategies is proposed.

1. Let us consider the game problem of bringing onto the prescribed set M a controlled object whose motion is described by the system of differential equations

$$\frac{dx}{dt} = A(t)\,x + B(t)\,u - C(t)\,v \tag{1.1}$$

Here $x = \{x_1, \ldots, x_n\}$ is an n-dimensional phase vector and u is an r-dimensional controlling force chosen by the first player; v is the s-dimensional control of the second player; $A(t)$, $B(t)$, $C(t)$ are continuous matrices of the appropriate dimensions. We assume that the instantaneous values of the controlling forces are subject to restrictions of the form

$$u \in U^*, \qquad v \in V^* \tag{1.2}$$

where U^* and V^* are convex bounded closed sets in the spaces E_r and E_s, respectively. We assume that the set M satisfies one of the following conditions:

1) the set M is a convex bounded set in E_n;

2) the set M is a linear subspace of the space E_n.

The payoff in the game under investigation is the quantity ϑ, i.e. the instant of arrival of the phase vector $x[t]$ at the prescribed set M. The first player seeks to minimize the quantity ϑ; the second player seeks to maximize it.

Such games problems are investigated in [1–11]; elements of the constructions employed in these papers are used in the present study.

We shall adhere to the following definitions of the players' strategies and of the solution of the system (1.1).

Definition 1.1. The "strategy" or "control" $U = U(x, t)$ $(V = V(x,t))$ is the law which places the $(n + 1)$-dimensional vector $\{x, t\}$ in correspondence with the convex closed set $U(x, t) \subset U^*$ $(V(x, t) \subset V^*)$. We assume below that the strategies are semicontinuous above by inclusion (*).

Definition 1.2. The "solution" of system (1.1) generated by some pair of strategies $U = U(x, t)$, $V = V(x, t)$ is any absolutely continuous vector function $x = x[t]$ which satisfies the conditions

$$x[t_0] = x^\circ, \qquad \frac{dx[t]}{dt} \in A(t)x[t] + B(t)U(x[t], t) - C(t)V(x[t], t) \tag{1.3}$$

for almost all $t \geqslant t_0$.

Here t_0 is the initial instant, x^0 is the initial value of the phase vector, and $Ax + BU - CV$ is the algebraic sum of the vector Ax and the sets BU and $-CV$, i.e. the sum of all vectors of the form $Ax + Bu - Cv$, where $\in U$, $v \in V$.

Definition 1.1 reflects the character of the information available to the players in the course of the game; specifically, each player knows at each instant $t = \tau \geqslant t_0$ the realized value of the phase vector $x = x[\tau]$. The players do not know the future behavior of the oponent for $t > \tau$. The existence of a solution of system (1.1) in the sense of Definition 1.2 can be proved by taking the limits of the corresponding broken Euler lines (e.g. see [10–12]).

We shall describe the construction of the strategy $V = V_\alpha(x,t)$ which guarantees to the second player a result which is arbitrarily close to being optimal. The conditions used in constructing the strategy $V_\alpha(x, t)$ are formulated in Sect. 3 below.

2. In this section we formulate an ancillary theorem which we shall use in Sect. 3 to construct the strategy $V = V_\alpha(x, t)$.

Let us be given some time interval $[t_0, \varkappa]$ during which the second player intends to

*) The set $U(x, t)$ is semicontinuous above by inclusion at the point $\{x^*, t^*\}$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any vector $u \in U(x, t)$, where $\| \{x,t\} - \{x^*, t^*\} \| \leqslant \delta$ there exists a vector $u^* \in U(x^*, t^*)$ such that $\|u - u^*\| \leqslant \varepsilon$.

avoid arrival of the point $x\ [t]$ at the set M. We denote by G a bounded set in E_n such that the condition $x\ [t] \in G$ for $t_0 \leqslant t \leqslant \varkappa$ is fulfilled for the solution $x = x\ [t]$ of system (1.1) corresponding to any summable controls $u = u\ (t)$, $v = v\ (t)$ which satisfy restrictions (1.2). Such a set G always exists.

Let us suppose that there exists a function $L = L\ (x,\ t)$ which satisfies the following requirements:

1) the function $L = L\ (x,\ t)$ is defined and continuously differentiable with respect to x and t in some open domain D ; the point $\{x^0,\ t_0\} \in D$.

2) If $\{x, t\} \in D$, then $x \bar{\in} M$.

3) Let $\Gamma_{a,b}$ the set of points $\{x, t\}$ for which $x \in G$, $t_0 \leqslant t \leqslant \varkappa - a$, $L\ (x,\ t) \leqslant b$. The third condition imposed on the function $L = L\ (x,\ t)$ is as follows: for any values of the parameters $a > 0$, $b \geqslant L\ (x^0,\ t_0)$ there exists a number $\varepsilon\ (a,\ b) > 0$ such that $\Gamma_{a,b}{}^{\varepsilon(a,b)} \subset D$, where Γ^ε is the ε-neighborhood of the set Γ (i.e. the set of points of the form $q_1 + q_2$, where $q_1 \in \Gamma$, $\|q_2\| \leqslant \varepsilon$, and where the symbol $\|q\|$ denotes the Euclidean norm of the vector q).

4) Let us consider the quantity

$$\Phi\,(x,\ t,\ u,\ v) = (\mathrm{grad}_x\, L\,(x,\ t))'\ \ A\,(t)\,x + B\,(t)\,u - C\,(t)\,v) + \frac{\partial L\,(x,\ t)}{\partial t} \qquad (2.1)$$

(where the prime denotes transposition) defined for the points $\{x, t\} \in D$. Since $\Phi\ (x,\ t,\ u,\ v)$ can be expressed as a sum of two terms, the first of which depends on u only and the second on v, we have

$$\min_v = \max_u \Phi\ (x,\ t,\ u,\ v) = \max_u \min_v \Phi\,(x,\ t,\ u,\ v) = \Phi^0\ (x,\ t)$$
$$(u \in U^*,\ \ v \in V^*) \qquad (2.2)$$

The final condition which we impose on $L\ (x,\ t)$ is that it satisfy the inequality

$$\Phi^0\ (x,\ t) \leqslant c, \quad \text{for } \{x,\ t\} \in D\ \cap\ (G \times\ [t_0,\varkappa]) \qquad (c = \mathrm{const}) \qquad (2.3)$$

We denote by $V^0\ (x,\ t)$ the set of vectors which ensure that

$$\max_v (\mathrm{grad}_x L\ (x,\ t))'\ C\ (t)\ v \quad \text{for} \quad v \in V^* \qquad (2.4)$$

(in the case $(\mathrm{grad}_x L\ (x,\ t))'C\ (t) = 0$ we set $V^0\ (x,\ t) = V^*$). We can show that if $\{x,\ t\} \in D$ the strategy $V^0\ (x,\ t)$ satisfies Definition 1.1.

The following statement is valid.

T h e o r e m 2.1. If these exists a function $L = L\ (x,\ t)$ satisfying conditions (1)–(4), then any solution of the system

$$dx/dt \in A\ (t)\ x + B\ (t)\ U\ (x,\ t) - C\ (t) V^0(x,\ t),\ x\ [t_0] = x^0 \qquad (2.5)$$

where $U\ (x,\ t)$ is an arbitrary permissible strategy defined for $\{x,\ t\} \in G \times\ [t_0,\varkappa]$, is continuable to the instant $t = \varkappa$; for any solution of system (2.5) we have $x\ [t] \bar{\in} M$ for all $t_0 \leqslant t < \varkappa$.

P r o o f. We know that the solution $x = x\ [t]$ of the system

$$dx\ /\ dt \in A\ (t)\ x + W\ (x,\ t), \qquad x\ [t_0] = x^0 \qquad (2.6)$$

is continuable to the instant $t = t^*$ as long as $\{x\ [t], t\} \in N$ for all $t_0 \leqslant t < t^*$, where N is the open set of points $\{x,\ t\}$ in which $W\ (x,\ t)$ is semicontinuous above by inclusion, convex, bounded, and closed. Hence, in order to show that any solution of system (2.5) is continuable to the instant $t = \varkappa$ we need merely show that

$$\{x\ [t], t\} \in D \quad \text{for} \quad t_0 \leqslant t < \varkappa \qquad (2.7)$$

for any solution $x = x[t]$ of system (2.5).

Let us assume that the opposite is true. Let $t = t_* < \varkappa$ be the instant when

$$\lim_{t \to t_* - 0} \{x[t], t\} = \{x^*, t_*\} \overline{\in} D \tag{2.8}$$

for some solution $x = x[t]$ of system (2.5).

We note that this solution satisfies the inclusion

$$\{x[t], t\} \in D \quad \text{for} \quad t_0 \leqslant t < t_* \tag{2.9}$$

We also note that by virtue of (2.1)–(2.4)

$$dL[t] / dt \leqslant c \tag{2.10}$$

for $t < t_*$ along the indicated solution in the case of absolutely continuous function $L[t] = L(x[t], t)$.

This means that

$$L(x[t], t) \leqslant L(x^0, t_0) + c(t - t_0) \quad \text{for} \quad t < t_* \tag{2.11}$$

Setting

$$a = \varkappa - t_*, \quad b = L(x^0, t_0) + c(\varkappa - t_0) \tag{2.12}$$

we obtain the corresponding number $\varepsilon = \varepsilon(a, b) > 0$ by way of condition (3). Let $t' < t_*$ be an instant such that

$$\| \xi[t'] - \xi^* \| \leqslant {}^1/_2\, \varepsilon \tag{2.13}$$

where $\xi[t'] = \{x[t'], t'\}$, $\xi^* = \{x^*, t_*\}$. Since $t' < t_*$, condition (3) together with (2.11), (2.12) imply that

$$S_\varepsilon(\xi[t']) \subset D \tag{2.14}$$

where $S_\varepsilon(\xi)$ is the ε-neighborhood of the point ξ. From (2.14), (2.13) we infer that $S_{\varepsilon/2}(\xi^*) \subset D$. This contradicts assumption (2.8). We have therefore established the validity of condition (2.7). By virtue of (2), condition (2.7) implies that $x[t] \in M$ for $t_0 \leqslant t < \varkappa$ for any solution of system (2.5). The statements of Theorem 2.1 have been proved.

3. Let us formulate the conditions whose fulfilment means that the function $L = L(x, t)$ can be constructed in such a way that the control $V = V^0(x, t)$, whose construction we have described in Sect. 2, guarantees a result arbitrarily close to the optimal one to the second player. Some remarks concerning the class of problems (1.1), (1.2) which satisfy the conditions below will be made at the close of the present section.

In constructing the function $L = L(x, t)$ we shall make use of the quantity $\varepsilon^0(x, t, \sigma)$ defined as follows.

Let $G_1(\tau, \sigma)$ be the set of points $g \in E_n$ for each of which there exists a permissible program control $u = u(t)$, $\tau \leqslant t \leqslant \sigma$, which brings the system

$$dx/dt = A(t)x + B(t)u \tag{3.1}$$

from the state $x(\tau) = 0$ to the position $x(\sigma) = g$. A "permissible program control" is a summable vector function $u = u(t)$, $\tau \leqslant t \leqslant \sigma$, which satisfies condition (1.2) almost everywhere. The set $G_1(\tau, \sigma)$ is called the "attainability domain" of object (3.1) which corresponds to the initial and final instants τ and σ, respectively.

In similar fashion we introduce the notion of the attainability domain $G_2(\tau, \sigma)$ of the controlled object whose motion is described by the system

$$dx/dt = A(t)x + C(t)v \tag{3.2}$$

Let M^ε be a closed ε-neighborhood of the set M, i.e. the set of points of the form $g_1 + g_2$, where $g_1 \in M$, $\| q_2 \| \leqslant \varepsilon$; $X(\sigma, t)$ is an $n \times n$ matrix which satisfies the equation $\partial X(\sigma, t) / \partial\sigma = A(\sigma) X(\sigma, t)$;

$X(t, t) = E$ is an identity matrix. We define $\varepsilon^0(x, t, \sigma)$ as the smallest value of the parameter ε for which we have the inclusion

$$G_2(\tau, \sigma) \subset G_1(\tau, \sigma) + X(\sigma, \tau)x - M^\varepsilon \tag{3.3}$$

As above, the expression on the right is an algebraic sum of the sets $G_1, -M^\varepsilon$ and of the vector Xx. The set $G(x, \tau, \sigma) = G_1(\tau, \sigma) + X(\sigma, \tau)x - M^\varepsilon$ can be interpreted as the attainability domain constructed for the final instant $\sigma \geqslant \tau$ for a controlled object whose motion is described by the system

$$dx^{(1)}/dt = A(t)x^{(1)} + B(t)u(t) - p\delta(\sigma - t), \ x(\tau) = x \tag{3.4}$$

In addition to the permissible program control $u = u(t)$ we have here a fictitious impulsive control $p\delta(\sigma - t)$, where $p \in M^\varepsilon$. The quantity $\varepsilon^0(x, t, \sigma)$ can be defined as in [10], i.e. as the smallest value of the parameter ε for which there is ε-absorption of one of the control processes by the other (in this case process (3.4) absorbs process (3.2)). The initial problem can be considered as the problem of game encounter of controlled objects (3.4), (3.2), where Eq. (3.4) describes the motion of the pursuer and Eq. (3.2) that of the pursued object (target); the initial position of the pursuer is $x^{(1)}[t_0] = x^0$; the initial position of the pursued object is $x[t_0] = 0$.

We can show that if M is a convex bounded closed set, then condition (3.3) is equivalent to the inequality

$$\min_l \{\rho_1(l, \tau, \sigma) + \rho(l) + (X(\sigma, \tau)x)'l + \varepsilon\|l\| - \rho_2(l, \tau, \sigma)\} \geqslant 0 \quad \text{for } \|l\| \leqslant 1 \tag{3.5}$$

where ρ_1, ρ_2, ρ are the supporting functions of the sets $G_1, G_2, -M$, respectively(*).

If M is a linear subspace of the space E_n, then condition (3.3) is equivalent to the inequality

$$\min_l \{\rho_1^*(l, \tau, \sigma) + \varepsilon\|l\| + l'(X(\sigma, \tau)x)^* - \rho^*_2(l, \tau, \sigma)\} \geqslant 0 \quad \text{for } \|l\| \leqslant 1 \tag{3.6}$$

Here ρ_1^*, ρ_2^* are the supporting functions of the sets G_1^*, G_2^*, i.e. of the projections of the sets G_1, G_2 onto the orthogonal complement of the subspace M which we denote by Q; g^* is the projection of the n-dimensional vector g onto the subspace Q; the dimensionality of the vector l is the same as that of the subspace Q.

The requirements imposed on the system (1.1), (1.2) can be formulated as follows.

Condition 3.1. For all permissible values of x, τ, σ satisfying the inequality $\varepsilon^0(x, \tau, \sigma) > 0$ the minimum in (3.5) (or 3.6)) is attained on the unique vector $l = l^0(x, \tau, \sigma)$.

The function $L = L(x, t)$ is defined by the formula

$$L(x, t) = \int_t^\varkappa \frac{1}{\varepsilon^0(x, t, \sigma)} d\sigma \qquad (\varkappa = \vartheta^\circ - \alpha) \tag{3.7}$$

Here $\alpha > 0$ is arbitrarily small and ϑ^0 is the smallest root of the equation

*) The supporting function $\rho_N(l)$ for an arbitrary convex closed set $N \subset E_n$ is defined by the formula $\rho_N(l) = \max l'x$ for $x \in N$, where $l = \{l_1, \ldots, l_n\}$ is an n-dimensional vector.

$$\varepsilon^0 (x^0, t_0, \sigma) = 0 \quad (\sigma > t_0) \tag{3.8}$$

On fulfilment of Condition 3.1 the function (3.7) satisfies requirements (1)–(3) formulated in Sect. 2.

Let us describe briefly the basic propositions used to verify this statement. We define the domain D as the set of points $\{x, t\}$ for which

$$\varepsilon^0 (x, t, \sigma) > 0 \quad \text{for} \quad t \leqslant \sigma \leqslant \varkappa \tag{3.9}$$

The differentiability of function (3.7) in the domain D follows from the differentiability with respect to x and t of the function $\varepsilon^0 = \varepsilon^0 (x, t, \sigma)$ in the neighborhood of those points $\{x, t, \sigma\}$ for which $\varepsilon^0 (x, t, \sigma) > 0$. The existence of the derivatives of the function $\varepsilon^0 (x, t, \sigma)$ can be demonstrated by applying Condition 3.1 to formula (3.5) (or (3.6)) (e.g. see [6, 13]).

The inclusion $\{x^0, t_0\} \in D$ is fulfilled by virtue of the definitions of the quantity $\varkappa$ and the domain D. The second condition formulated in Sect. 2 follows in this case from the definitions of the quantity $\varepsilon^0 (x, t, \sigma)$ and the domain D.

The third requirement is also fulfilled for the function $L = L (x, t)$ defined by (3.7). This can be verified by means of the inequality

$$| \varepsilon^0 (x, t, \sigma') - \varepsilon^0 (x, t, \sigma'')| \leqslant k |\sigma' - \sigma''| \quad \text{for} \quad \{x, t\} \in G \times [t_0, \varkappa] \tag{3.10}$$

$$\delta', \delta'' \leqslant \varkappa$$

which follows directly from the definition of the quantity $\varepsilon^0 (x, t, \sigma)$.

Thus, function (3.7) satisfies conditions (1)–(3) of Sect. 2 provided that Condition 3.1 is fulfilled. We also assume that system (1.1) and restrictions (1.2) are such that the following additional condition is satisfied.

Condition 3.2. Function (3.7) satisfies inequality (2.3) for $\varkappa < \vartheta^0$.

If Conditions 3.1 and 3.2 are fulfilled for the game problem under consideration, then function (3.7) satisfies all of the conditions formulated in Sect. 2. This means, by virtue of Theorem 2.1, that we can construct a strategy $V = V_\alpha (t, x)$ defined by relations (2.4), (3.7) which guarantees to the second player that the point $x = x [t]$ will arrive at M not earlier than at the instant $t = \varkappa = \vartheta^0 - \alpha$, where $\alpha > 0$ is arbitrarily small. We note that fulfilment of Condition 3.1 means that it is possible to construct a control $U^0(x, t)$ which guarantees to the first player the arrival of object (1.1) at M by the instant $t = \vartheta^0$. The construction of this control $U^0(x, t)$ described for pursuit problems in [9, 10] is wholly applicable to our game problem. This can be verified without going through the arguments of [9, 10], since, as we noted above, the problem under consideration here can be interpreted as the problem of game encounter of objects (3.2) and (3.4).

Thus, fulfilment of Conditions 3.1 and 3.2 means that there exist strategies $U^0 (x, t)$, $V_\alpha (x, t)$ of which the first guarantees the arrival of the phase vector $x = x [t]$ at M for any strategy $V (x, t)$ not later than at the instant $t = \vartheta^0$; the second strategy $V_\alpha (x, t)$ guarantees a postponement in the arrival of the trajectory $x = x [t]$ at the set M until the instant $\vartheta^0 - \alpha$ (where α is arbitrarily small) for any choice of the strategy $U (x, t)$. The strategy $U^0 (x, t)$ is therefore the first player's optimal strategy, and the strategy $V_\alpha (x, t)$ guarantees to the second player a result $\vartheta^0 - \alpha$ arbitrarily close (for small α) to the optimal result equal to the instant ϑ^0.

Effective verification of Conditions 3.1 and 3.2 in the general case is difficult. Let us formulate the sufficient restrictions which must be imposed on the coefficients of system (1.1) and on the sets U^*, V^*, M in order to ensure fulfillment of Conditions 3.1 and 3.2.

1) The sets

$$U_*(\sigma, \tau) = X(\sigma, \tau) B(\tau) U^*, \quad V_*(\sigma, \tau) = X(\sigma, \tau) C(\tau) V^*$$

satisfy the condition

$$U_*(\sigma, \tau) = V_*(\sigma, \tau) + W(\sigma, \tau) \tag{3.11}$$

where $W(\sigma, \tau)$ is some convex set.

2) For any vector $u \in U^*$ there exists a vector $v \in V^*$ such that

$$X(\sigma, \tau)(B(\tau) u - C(\tau) v) \in W(\sigma, \tau) \quad \text{for} \quad \tau \leqslant \sigma \leqslant \varkappa \tag{3.12}$$

Verification of the above relations is simpler than the verification of Conditions 3.1 and 3.2. In particular, conditions (3.11) and (3.12) are satisfied by the control example discussed in [3], and also by those game problems (1.1),(1.2) which can be interpreted as problems on the games encounter of monotype objects.

We note that relations (3.11),(3.12) with the conditions of [7] under which the control constructed in [4] is optimal. If the equation $\varepsilon^0(x^0, t_0, \sigma) = 0$ has no solution, then the above construction guarantees to the second player a postponement of the arrival of the phase point at the manifolds for any arbitrarily long time $\varkappa$.

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ON CONTROL PROBLEMS WITH RESTRICTED COORDINATES

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The approach developed in monograph [1] is used to consider the problem of control for a linear system with bounded phase coordinates. The properties of the solutions and of the boundary conditions for the corresponding associated system are discussed. Additional information is obtained about the Lagrange coefficients; this information can be used to reduce solutions of the initial multidimensional problem to the minimization of a function of a finite number of variables.

1. Formulation of the problem. Let us consider the controlled motion described by the equation

$$dx/dt = A(t)\,x + B(t)\,u + w(t) \tag{1.1}$$

Here the vector x is n-dimensional, the control u is r-dimensional, and the matrices $A(t)$, $B(t)$ and the perturbation n-vector $w(t)$ are continuous.

Problem 1.1. We are given system (1.1), boundary conditions $x(t_\alpha) = x_\alpha$, $x(t_\beta) = x_\beta$, and the restrictions

$$\text{vrai}\max_t |u_j(t)| \leqslant \nu_j, \quad t_\alpha \leqslant t \leqslant t_\beta \qquad (j = 1, \ldots, r) \tag{1.2}$$

on the control $u \in U$, and

$$|x_k(t)| \leqslant f_k(t) \qquad (k = 1, \ldots, m) \tag{1.3}$$

on the coordinates $x(t) \in X(t)$.

The functions $f_k(t)$ are absolutely continuous and positive. We are to bring system (1.1) from x_α to x_β in the minimum time $t_\beta{}^0 - t_\alpha$ under restrictions (1.2), (1.3).

2. The solvability conditions. The maximum principle. The result of the present section is valid for all closed convex restrictions $u \in U$, $x \in X$ on the instantaneous values of the controls and of the first m coordinates, provided that zero is an interior point of both U and X.

Let us assume that t_β is fixed and consider the following moment calculation problem:

$$\begin{aligned} &\int h_j(t_\beta, \tau)\, u(\tau)\, d\tau = c_{j\beta} && (j = 1, \ldots, n) \\ &\int h_k(t_i, \tau)\, u(\tau)\, d\tau + z_k{}^{(i)} = c_{ki} && (k = 1, \ldots, m) \end{aligned} \tag{2.1}$$

for

$$u \in U, \quad z^{(i)} \in Z(t_i) \qquad (Z = -X) \tag{2.2}$$

Here $\{t_i\}$, the set of points (e.g. of the form $t_\alpha + i(N)(t_\beta - t_\alpha)\, 2^{-N}$, where $i(N) \leqslant 2^N$, $N = 1, 2, 3, \ldots$), is dense everywhere in the segment (*) $[t_\alpha, t_\beta]$;

*) See Note at the bottom of next page.